Question

What is the expected number of spins in double-zero roulette to see five reds or five blacks in a row?

Answer

123929248 / 3779136 = 3872789/118098 =~ 32.79301089 spins.

Solution

There are five possible states the player can be in. Let's define them as follows:

a = Expected future spins from the initial state and after any green.

b = Expected future spins after a streak of one red or black.

c = Expected future spins after a streak of two reds or two blacks.

d = Expected future spins after a streak of three reds or three blacks.

e = Expected future spins after a streak of four reds or four blacks.

As a reminder, a double-zero roulette wheel has 18 reds, 18 blacks, and 2 greens.

From state a there is a probability of 36/38 of advancing to state b (with a red or black on the next spin) and 2/38 of staying in state a (with a green). We can express that mathematically as:

$$a = 1 + (2/38)a + (36/38)b$$

From state b there is a probability of 18/38 of advancing to state c (with whatever the last spin was), 18/38 of staying in state b (with the opposite of the last spin) and 2/38 of regressing to state a (with a green). We can express that mathematically as:

$$b = 1 + (2/38)a + (18/38)b + (18/38)c$$

From state c there is a probability of 18/38 of advancing to state d (with whatever the last spin was), 18/38 of going back to state b (with the opposite of the last spin) and 2/38 of regressing to state a (with a green). We can express that mathematically as:

$$c = 1 + (2/38)a + (18/38)b + (18/38)d$$

From state d, there is a probability of 18/38 of advancing to state e (with whatever the last spin was), 18/38 of going back to state b (with the opposite of the last spin) and 2/38 of regressing to state a (with a green). We can express that mathematically as:

$$d = 1 + (2/38)a + (18/38)b + (18/38)e$$

From state e, there is a probability of 18/38 of successfully completing the goal (with whatever the last spin was), 18/38 of going back to state b (with the opposite of the last spin) and 2/38 of regressing to state a (with a green). We can express that mathematically as:

$$e = 1 + (2/38)a + (18/38)b$$

Let's multiply all those equations by 38, to get rid of the factions:

$$38a = 38 + 2a + 36b$$

$$38b = 38 + 2a + 18b + 18c$$

$$38c = 38 + 2a + 18b + 18d$$

$$38d = 38 + 2a + 18b + 18e$$

$$38e = 38 + 2a + 18b$$

Let's reorganize those equations to get them ready to put in a matrix:

$$36a - 36b = 38$$

$$2a - 20b + 18c = 38$$

$$2a + 18b - 38c + 18d = 38$$

$$2a + 18b - 38d + 18e = 38$$

$$2a + 18b - 38e = 38$$

At this point, we have five equations and five unknowns, so we have enough to solve for all five variables. However, let's do it the elegant way with matrix algebra.

Here are the five equations in matrix form:

36	-36	0	0	0	-38
2	-20	18	0	0	-38
2	18	-38	18	0	-38
2	18	0	-38	18	-38
2	18	0	0	-38	-38

Let matrix X be the first five columns:

36	-36	0	0	0
2	-20	18	0	0
2	18	-38	18	0
2	18	0	-38	18
2	18	0	0	-38

Let matrix Y be the same as X, except column six in the original matrix in column 1 of matrix X:

-38	-36	0	0	0
-38	-20	18	0	0
-38	18	-38	18	0
-38	18	0	-38	18
-38	18	0	0	-38

a = determinant(Y)/determinant(X).

In Excel, you can easily find the determinant of a matrix with the function MDETERM(matrix range). Note that older versions of Excel have a different process, involving the ALT key, if I remember correctly.

Doing so gives us:

determinant(X) = 3779136

determinant(Y) = 123929248

a = 123929248 / 3779136 = 3872789/118098 =~ 32.79301089