Problem: A player plays video poker. The probability of hitting a royal flush on any hand is p. What is the probability that in n hands there will be no "drought" of d consecutive hands without a royal flush?

Solution 1. Assume that the player plays at a constant speed, and choose the unit of time such that expected number of royal flushes per unit time is 1.

Let X_i be the time of the ith royal flush, we can model this with the jump times of a homogeneous Poisson process with parameter 1. We can consider X_i $(i = 1, 2, 3, \ldots)$ as a sequence of random variables such that X_1, $X_2 - X_1$, $X_3 - X_2$, etc., are independent exponentially distributed random variables with parameter 1.

The question can be restated in these term as follows. Given k and x, what is the probability that X_1, $X_2 - X_1$, …, $X_m - X_{m-1}$ and $x - X_m$ are all at most k, where X_m is the largest m such that $X_m < x$? (In the original question, $d = 200000$, $n = 1000000$, $p = 1/40391$, so $k = 200000/40391$, $x = 1000000/40391$.)

Let this probability be $f(x)$. Clearly $f(x) = 1$ if $0 \le x \le k$. If $x > k$, then consider X_1. We must have $X_1 \le k$, and then $X_i - X_{i-1}$ $(i = 2, 3, \ldots, m)$ and $x - X_m$ must all be at most k, which has probability $f(x - X_1)$. This gives the following equation for $f(x)$ if $x > k$:

$$f(x) = \int_0^k e^{-t} f(x - t)\, dt. \tag{1}$$

By using the Dirac δ-function, this can be rewritten in a form valid for all x:

$$f(x) = \int_0^k e^{-t} (f(x - t) + \delta(x - t))\, dt. \tag{2}$$

(The δ-function is not a "real" function, it is a so-called generalized function or distribution. It can be considered as a notational convenience, it is defined by the property that $\int_{-\infty}^{\infty} g(x)\delta(x)\, dx = g(0)$ for any function g. In the above equation it adds e^{-x} to the right-hand side if $0 \le x \le k$ to make it correct for all $x > 0$.)

The Laplace transform $\mathcal{L}(g)$ of a function $g(t)$ defined on $[0, \infty)$ is

$$\mathcal{L}(g)(s) = \int_0^{\infty} g(t) e^{-st}\, dt.$$

Let $h(x) = e^{-x}$ if $0 \leq x \leq k$, and 0 otherwise. Then the right-hand side of (2) is the convolution of $h(x)$ with $f(x) + \delta(x)$, and it is a well-known property of Laplace transforms that the Laplace transform of the convolution is the product of the Laplace transforms of the factors.. Hence we obtain

$$\mathcal{L}(f) = \mathcal{L}(h)(\mathcal{L}(f) + \mathcal{L}(\delta)).$$

$\mathcal{L}(\delta) = 1$ and $\mathcal{L}(h) = \dfrac{1 - e^{-k(1+s)}}{1+s}$. We can solve for $\mathcal{L}(f)$,

$$\mathcal{L}(f) = \frac{\mathcal{L}(h)}{1 - \mathcal{L}(h)} = \frac{-1 + e^{k(1+s)}}{1 + e^{k(1+s)}s}. \tag{3}$$

The middle expression can be expanded into a power series, so $\mathcal{L}(f) = \sum_{r=1}^{\infty} [\mathcal{L}(h)]^r$. The Laplace transform converts convolutions to products, therefore by inverting it we get $f(x) = \sum_{r=1}^{\infty} *^r h(x)$, where $*^r h(x)$ is the r-fold convolution of h with itself. This seems to be the most explicit form of $f(x)$ but it does not appear to be useful for calculations.

Instead of an exact form we need to find an approximation. The asymptotic behaviour of $f(x)$ as $x \to \infty$ is determined by the poles of $\mathcal{L}(f)$, the places where $\mathcal{L}(f)$ is not defined because the denominator is 0. Both real and complex values need to be considered. Near a real pole $s = a$, $\mathcal{L}(f)$ behaves like $A/(s-a)$ for some constant A, and this corresponds to a term Ae^{ax} in $f(x)$. Complex poles come in conjugate pairs, $b \pm ci$, and their joint effect is a term of the form $e^{bx}(\alpha \cos(cx) + \beta \sin(cx))$.

We need to find where the denominator of $\mathcal{L}(f)$ vanishes. $1 + e^{k(1+s)}s = 0$ cannot be solved for s using standard functions. It can be rearranged to the form

$$kse^{ks} = -ke^{-k}. \tag{4}$$

$s = -1$ is clearly a solution, but there is no pole there unless $k = 1$, because the numerator of $\mathcal{L}(f)$ also vanishes. There is a function called productlog, sometimes denoted by W, which is the inverse of ze^z, so that it satisfies $z = W(z)e^{W(z)}$. Let

$$a = \frac{W(-ke^{-k})}{k},$$

where we are using the non-standard convention that out of the two possible real values of W we take the one not equal to $-k$, unless $k = 1$, in which case the only possible value is -1. $s = a$ and $s = -1$ are the only real solutions of $1 + e^{k(1+s)}s = 0$ and $\mathcal{L}(f)$ has a pole at $s = a$.

If $k = 1$ then $a = -1$, if $k \neq 1$, then k can be expressed in terms of a as

$$k = \frac{-\ln(-a)}{1+a}.$$

For practical purposes a can be calculated by solving this equation numerically to the required accuracy.

We now claim that $f(x)$ is asymptotically equal to Ae^{ax} for some A. The only real pole of $\mathcal{L}(f)$ is at $s = a$, if the dominant term of $f(x)$ came from a pair of complex poles, then it would be of the form $e^{bx}(\alpha \cos(cx) + \beta \sin(cx))$, but since it oscillates and changes sign, it cannot be the dominant term in $f(x)$ which is positive and monotonically decreasing. (This might also follow from certain properties of the complex values of the W function, which may be known to people who deal with it more often than I do.)

The coefficient A can be determined as

$$A = \lim_{s \to a}(s-a)\mathcal{L}(f)(s) = \lim_{s \to a} \frac{(s-a)(-1+e^{k(1+s)})}{1+e^{k(1+s)}s}.$$

If $k \neq 1$, L'Hôpital's rule gives

$$A = \frac{(1+a)^2}{1+a-a\ln(-a)} = \frac{1+a}{1+ak},$$

while if $k = 1$, L'Hôpital's rule needs to be applied twice to get $A = 2$. Therefore if $k \neq 1$,

$$f(x) \approx \frac{1+a}{1+ak}e^{ax},$$

while if $k = 1$,

$$f(x) \approx 2e^{-x}.$$

Summary

Let p be the probability of the royal flush, d the length of the "drought", n the total number of hands played.

1. Set $k = dp$, $x = np$.
2. If $k = 1$ then let $a = -1$, otherwise find a such that $k = -\ln(-a)/(1+a)$. (a is a negative number, if $k > 1$ then $-1 < a < 0$, if $k < 1$ then $a < -1$, and a needs to be calculated to high accuracy.)
3. If $k = 1$, then let $A = 2$, otherwise let $A = (1 + a)/(1 + ak)$.
4. The probability of no "drought" of length d in n hands is approximately Ae^{ax}.

In the original problem, $k = 200000/40391 = 4.9516$, $x = 1000000/40391 = 24.758$. Hence $a = -0.00733363$, $A = 1.03007$, and the probability of no "drought" is $f(x) \approx 0.859042$, and the probability of there being a"drought" is $1 - f(x) \approx 0.140958$.

Solution 2.

Let b_n be the probability that in n hands of video poker there is no royal flush "drought" of length d. $b_0 = b_1 = \ldots = b_{d-1} = 1$, and for $n \geq d$, $b_n = p\sum_{i=1}^{d}(1-p)^{i-1}b_{n-i}$, this is the discrete equivalent of (1).

Let

$$\phi(x) = x^d - p\sum_{i=1}^{d}(1-p)^{i-1}x^{d-i} = \frac{x^{d+1} - x^d + p(1-p)^d}{x - (1-p)}.$$

It can be verified that $\phi(x)$ has no multiple roots, so the exact formula is of the form $b_n = \sum_{i=1}^{d}\alpha_i x_i^n$, where the x_i $(i = 1, 2, \ldots, d)$ are the roots of the characteristic equation $\phi(x) = 0$.

The coefficients can be determined by using Joshua Green's idea from `http://www.princeton.edu/~jvgreen/RandomEvent.pdf` as

$$\alpha_i = \frac{(1-p)^d}{\phi'(x_i)(1-x_i)}.$$

$\phi(x)$ has a root of maximum modulus close to but slightly less than 1, call it x_1. For large n, all the terms in b_n apart from $\alpha_1 x_1^n$ are negligible, and

$$b_n \approx \alpha_1 x_1^n = \frac{(1-p)^d x_1^n}{\phi'(x_1)(1-x_1)}.$$

Using the numbers in the original question, $x_1 = 0.9999998184456574$ and $\alpha_1 = 1.03007$, hence $b_{1000000} = 1.03007 \times 0.9999998184456574^{1000000} = 0.859050$ is the probability of no "drought" of length 200000 in 1000000 hands of video poker, in excellent agreement with the previous solution.

While this second method may give a more accurate result in theory, calculating x_1 to sufficient accuracy is not simple.